TEMPERATURE FIELD IN A NONSTATIONARY ADIABATIC GAS FLOW

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The conditions for the onset of thermal nonequilibrium in a nonstationary one-dimensional gas flow with a small mechanical nonequilibrium are analyzed.

Under certain conditions, a nonstationary adiabatic gas flow becomes a thermally nonequilibrium flow. This effect can be used to design simple highly efficient valveless reciprocating heat engines and refrigerators [1]. It can also have a substantial effect on the characteristics of various valveless heat engines and refrigerators with nonstationary working-fluid flows, such as heat engines employing the Stirling cycle [2].

Among the characteristic features of the nonstationary gas flows in such thermal machines are small gas-particle velocities and accelerations ($w/a \sim 5 \cdot 10^{-2}$, $Dw/dt \sim 1 \cdot 10^3$ m/sec^2), a negligible (compared to the duration of the machine's working cycle) transit time of rarefaction and compression waves in the flow, and a close to one-dimensional nature of the flow. It has been also shown by Khaskind [3] that the pressure gradient in the flow is small and that effects associated with the heat conductivity and viscosity of the gas are negligible.

Accordingly, we shall examine the one-dimensional nonsteady flow of an inviscid thermally nonconducting perfect gas in a thermally nonconducting tube of constant cross section, open at both ends (Fig. 1). The gas parameters in the flow are defined by the following system of equations:

$$\frac{Dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1a}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho w)}{\partial x} = 0, \tag{1b}$$

$$\frac{D}{dt}\left(\frac{p}{\rho^n}\right) = 0, \tag{1c}$$

$$\frac{p}{\rho} = RT. \tag{1d}$$

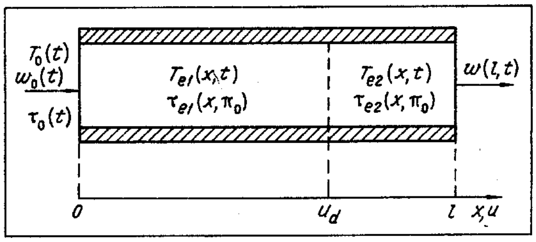

Fig. 1. Schematic diagram of the gas flow.

By simple transformations, ρ can be eliminated from Eq. (1), and the equations can be written in a form more convenient for analysis:

$$\frac{Dw}{dt} = -\frac{a^2}{n}\frac{\partial \pi}{\partial x}, \tag{2a}$$

$$\frac{D\pi}{dt} = -n\frac{\partial w}{\partial x}, \tag{2b}$$

$$\frac{D}{dt}(\tau - \pi) = 0, \tag{2c}$$

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where

$$\pi = \ln p;\ \tau = \frac{n}{n-1} \ln T;\ a^2 = nRT.$$

We shall obtain solutions of the system of equations (2), which correspond to small values of $\partial\pi/\partial x$ and to negligible transit times of rarefaction and compression waves in the flow. In this practically important case, we may set

$$w(x, t) = w_e(x, t) + w'(x, t), \tag{3a}$$

$$\tau(x, t) = \tau_e(x, t) + \tau'(x, t), \tag{3b}$$

$$\pi(x, t) = \pi_e(t) + \pi'(x, t), \tag{3c}$$

where the small quantities w', τ', π' characterize the disturbance introduced by a mechanical nonequilibrium of the flow, and w_e, τ_e correspond to a mechanical equilibrium flow ($\partial\pi/\partial x = 0$) at a gas pressure of $p_e = \exp\pi_e$.

The functions (3) must satisfy the boundary conditions*

$$\tau(0, t) = \tau_0(t), \tag{4a}$$

$$\pi(0, t) = \pi_0(t), \tag{4b}$$

$$w(0, t) = w_0(t) \tag{4c}$$

and the initial condition†

$$\tau(x, t_s) = \tau_s(x). \tag{4d}$$

By substituting expressions (3) into Eq. (2) and eliminating terms of higher-order smallness, we obtain a system of quasi-linear differential equations which define the required functions

$$\frac{\partial w_e}{\partial x} + \frac{1}{n}\frac{d\pi_e}{dt} = 0, \tag{5a}$$

$$\frac{\partial\tau_e}{\partial t} + w_e\frac{\partial\tau_e}{\partial x} - \frac{d\pi_e}{dt} = 0, \tag{5b}$$

$$\frac{\partial\pi'}{\partial x} + \frac{n}{a^2}\left(\frac{\partial w_e}{\partial t} + w_e\frac{\partial w_e}{\partial x}\right) = 0, \tag{5c}$$

$$\frac{\partial w'}{\partial x} + \frac{1}{n}\left(\frac{\partial\pi'}{\partial t} + w_e\frac{\partial\pi'}{\partial x}\right) = 0, \tag{5d}$$

$$\frac{\partial}{\partial t}(\tau' - \pi') + w_e\frac{\partial}{\partial x}(\tau' - \pi') = 0. \tag{5e}$$

Since the quantities π', w', τ' may be treated as small corrections to π_e, w_e, τ_e owing to a mechanical nonequilibrium of the flow, we have

$$w'(0, t) = 0, \tag{6a}$$

$$\pi'(0, t) = 0, \tag{6b}$$

$$\tau'(0, t) = 0 \tag{6c}$$

and correspondingly

$$w_e(0, t) = w_0(t), \tag{7a}$$

$$\tau_e(0, t) = \tau_0(t), \tag{7b}$$

$$\tau_e(x, t_s) = \tau_s(x), \tag{7c}$$

$$\pi_e(t) = \pi_0(t). \tag{7d}$$

*The system of coordinates is selected such that the value $x = 0$ corresponds to the tube end that serves as the gas inlet.

†The solutions of (3) are independent of the initial condition $w(x, t_s) = 0$ [3], because of the negligible propagation time of rarefaction and compression waves in the flow.

Together with the uniqueness conditions (7), the Eqs. (5a), (5b) define the principal terms of the solutions – the functions $w_e(x, t)$, $\tau_e(x, t)$, $\pi_e(t)$. With the aid of the remaining equations of system (5), $w_e(x, t)$, $\tau_e(x, t)$ can be determined from the uniqueness conditions (6) and the function $\pi'(x, t)$, $w'(x, t)$, $\tau'(x, t)$.

In the integration of Eqs. (5a), (5b), it is convenient to change the independent variable t for π_0 and w_e for v_e

$$v_e = w_e \left(\frac{d\pi_0}{dt} \right)^{-1}.$$

In this case, Eqs. (5a), (5b) take the form

$$\frac{\partial v_e}{\partial x} + \frac{1}{n} = 0, \tag{8a}$$

$$\frac{\partial \tau_e}{\partial \pi_0} + v_e \frac{\partial \tau_e}{\partial x} - 1 = 0, \tag{8b}$$

while the conditions (7) may be written as

$$\tau_e(0, \pi_0) = \tau_0(\pi_0), \tag{9a}$$

$$v_e(0, \pi_0) = v_0(\pi_0), \tag{9b}$$

$$\tau_e(x, \pi_s) = \tau_s(x), \tag{9c}$$

where

$$\pi_s = \pi_0(t_s).$$

From Eq. (8a), it follows that v_e and w_e are linear functions of the coordinate

$$v_e(x, \pi_0) = v_0(\pi_0) - \frac{x}{n}.$$

The field of τ_e values is described by a first-order linear partial differential equation obtained by substituting (10) into (8b)

$$\frac{\partial \tau_e}{\partial \pi_0} + \left[v_0(\pi_0) - \frac{x}{n} \right] \frac{\partial \tau_e}{\partial x} - 1 = 0. \tag{11}$$

The system of characteristic equations that correspond to Eq. (11) can be represented in the form

$$d\pi_0 = \frac{dx}{v_0(\pi_0) - \frac{x}{n}} = d\tau_e. \tag{12a}$$

System (12a) possesses two independent integrals

$$C_1 = x \exp \frac{\pi_0}{n} - \int v_0(\pi_0) \exp \frac{\pi_0}{n} \, d\pi_0, \tag{12b}$$

$$C_2 = \tau_e - \pi_0. \tag{12c}$$

The general solution of Eq. (11) is an arbitrary function of the integrals C_1, C_2 [4]

$$F(C_1, C_2) = 0. \tag{13}$$

A solution of (11) is uniquely defined if the uniqueness condition – the curve through which the integral surface (13) must pass – is given. The problem under consideration contains two uniqueness conditions: an initial condition (9c) and a boundary condition (9a). In the general case, to these conditions correspond two different integral surfaces (13) $F_1(\tau_e, x, \pi_0) = 0$ and $F_2(\tau_e, x, \pi_0) = 0$. One of these surfaces, together with the function $\tau_{e2}(x, \pi_0)$ that corresponds to it, satisfies the initial condition (9c) and describes the field of τ_e values in the gas initially contained in the tube. The other integral surface, together with the corresponding function $\tau_{e1}(x, \pi_0)$ satisfies the boundary (9a) and describes the field of τ_e values in the gas admitted to the tube through the section x = 0. Thus, the field of τ_e values is described by two functions:

a) $x \leq u_d(\pi_0)$

$$\tau_e(x, \pi_0) = \tau_{e1}(x, \pi_0);$$

b) $x \geq u_0(\pi_0)$

$$\tau_e(x, \pi_0) = \tau_{e2}(x, \pi_0).$$

Here, $u_d(\pi_0)$ is the coordinate of particles situated in the cross section $x = 0$ at an initial pressure π_s.

From relation (10), it follows that, correct to small terms,

$$\frac{du}{d\pi_0} + \frac{u}{n} - v_0(\pi_0) = 0. \tag{14a}$$

By integrating the differentiation equation (14a) for the initial condition $u(\pi_s) = 0$, we obtain

$$u_d(\pi_0) = \exp\left(-\frac{\pi_0}{n}\right) \int\limits_{\pi_s}^{\pi_0} v_0(\xi) \exp\frac{\xi}{n}\, d\xi. \tag{14b}$$

In the general case, a steady temperature discontinuity surface can exist in the cross section $x = u_d(\pi_0)$. From thermodynamic considerations it follows that for $\tau_s(0) - \tau_0(\pi_s) = 0$ the discontinuity is weak and is experienced only by the derivatives, while for $\tau_s(0) - \tau_0(\pi_s) \neq 0$ we have a strong discontinuity, in which case a jump

$$\Delta\tau = \tau_s(0) - \tau_0(\pi_s)$$

travels through the tube.

A method proposed in [4] will be used for obtaining a solution of Eq. (11), which corresponds to the uniqueness condition (9a). By substituting the boundary condition (9a) into the relations (12b) and (12c), we obtain

$$C_1 = -J(\pi_0), \tag{15a}$$
$$C_2 = \tau_0(\pi_0) - \pi_0, \tag{15b}$$

where

$$J(\pi_0) = \int v_0(\pi_0) \exp\frac{\pi_0}{n}\, d\pi_0.$$

By solving the Eq. (15b) with respect to π_0

$$\pi_0 = \varphi(C_2) \tag{15c}$$

and substituting (15c) into (15a), we obtain

$$C_1 + J[\varphi(C_2)] = 0. \tag{16a}$$

From expressions (16a), (12b), (12c) it follows that

$$x \exp\frac{\pi_0}{n} + \int\limits_{\pi_0}^{\varphi(\tau_{e1}-\pi_0)} v_0(\xi) \exp\frac{\xi}{n}\, d\xi = 0. \tag{16b}$$

Expression (16b) defines the function $\tau_{e1}(x, \pi_0)$ as an implicit function.

The solution of Eq. (11) that corresponds to the initial condition (9c) may be represented in the form

$$\tau_{e2}(x, \pi_0) = \tau_s(x_s) + \pi_0 - \pi_s, \tag{17a}$$

where the auxiliary variable x_s is defined as follows:

$$x_s = x \exp\frac{\pi_0 - \pi_s}{n} - \int\limits_{\pi_s}^{\pi_0} v_0(\xi) \exp\frac{\xi - \pi_s}{n}\, d\xi. \tag{17b}$$

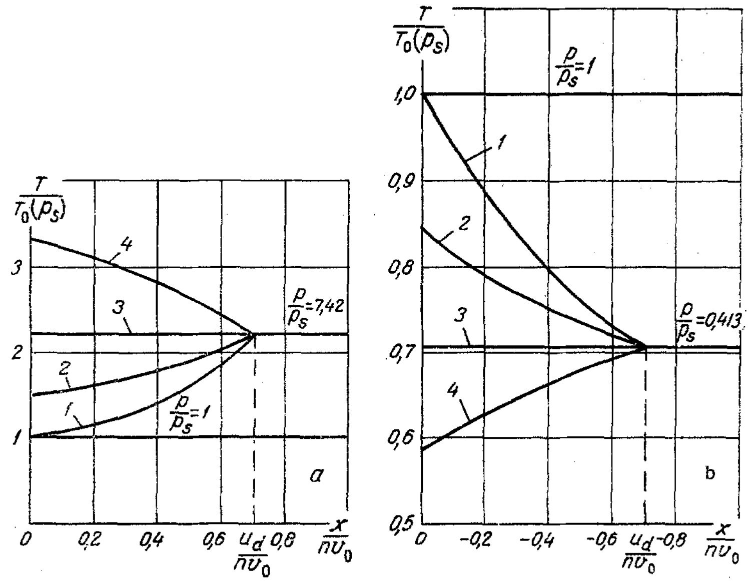

Fig. 2. Temperature field in a nonstationary gas flow: a) for increasing pressure; b) for decreasing pressure. 1) ψ = 0; 2) ψ) 0.5; 3) ψ = 1.0; 4) ψ = 1.5.

For evaluating the values of w', τ', π' it is advantageous to examine the simple case where $d^2\pi_0/dt^2 = 0$, $dv_0/d\pi_0 = 0$, $d\tau_0/d\pi_0 = 0$, $u_d(\pi_0) \geq l$. Here, the solutions of Eqs. (5c), (5d), (5e) that correspond to the uniqueness conditions (6) may be written in the form

$$\tau' = \pi' = \frac{n}{n+1}[M_e^2(0) - M_e^2(x)], \tag{18a}$$

$$w' = \frac{1}{n+2}[M_e^2(x)\,w(x) - M_e^2(0)\,w(0)], \tag{18b}$$

where

$$M_e = \frac{w_e}{a}.$$

Expressions (18a), (18b) show that the disturbances τ', w', π' caused by a mechanical nonequilibrium of the flow have an order of smallness of M_e^2. For conditions characteristic of thermal machines ($w/a \sim 5 \cdot 10^{-2}$), this leads to extremely small (relative values below 10^{-3}) corrections to $T_e = \exp[(n-1)/n]\tau_e$, p_e, w_e. Therefore, for nonstationary gas flows encountered in thermal machines one may accept, with an accuracy completely satisfactory for engineering purposes, the following expressions:

$$T = \exp\frac{n-1}{n}\tau_e, \tag{19a}$$

$$w = w_e. \tag{19b}$$

Taking these expressions into account, let us analyze the conditions for the onset of thermal nonequilibrium in gas flows of thermal machines. Analysis of expression (16b) yields the most significant results, since beginning with the moment at which $u_d(\pi_0)$ becomes equal to l, the temperature field in the flow is described solely by the function $\tau_{e1}(x, \pi_0)$.

From relations (16b) and (16c) it follows that

$$\frac{\partial\tau_{e1}}{\partial x} = \left[1 - \frac{d\tau_0(\pi_0)}{d\pi_0}\right] \frac{\exp\frac{1}{n}[\pi_0 - \varphi(\tau_{e1} - \pi_0)]}{v_0[\varphi(\tau_{e1} - \pi_0)]} \tag{20}$$

Since, in correspondence with (19a)

$$\frac{\partial T}{\partial x} = \frac{n-1}{n} T \frac{\partial \tau_e}{\partial x},$$

expression (20) makes it possible to determine the conditions for the thermal nonequilibrium of the flow for $x \le u_d(\pi_0)$. This expression shows that the onset of thermal nonequilibrium is defined by the temperature boundary conditions $\tau_0(\pi_0)$.

If the gas flows to the tube from an adiabatic cylinder, then $d\tau_0(\pi_0)/d\pi_0 = 1$ and the flow is in thermal equilibrium: $\partial\tau_{e1}/\partial x = 0$; at the same time, $\partial\tau_{e1}/\partial\pi_0 = 1$.*

If the gas directly in front of the tube releases heat when the pressure is increased, and heat is supplied to the gas when the pressure drops, we have $d\tau_0(\pi_0)/d\pi_0 < 1$. It is noteworthy that this case is quite typical for heating and refrigerating units; a positive temperature gradient ($\partial\tau_{e1}/\partial x > 0$) is established in the tube when the pressure increases ($v_0 > 0$), while a negative pressure gradient ($\partial\tau_{e1}/\partial x < 0$) is established when the pressure drops ($v_0 < 0$). It is characteristic that for $d\tau_0(\pi_0)/d\pi_0 < 1$ also $\partial\tau_{e1}/\partial\pi_0 < 1$.

If the gas flows into the tube at a constant temperature and in addition $dv_0/d\pi_0 = 0$, then for $x \le u_d(\pi_0)$ a stationary temperature field

$$T = T_0 \left(1 - \frac{x}{nv_0}\right)^{1-n} \tag{21}$$

will be present in the flow.

Finally, let us examine the case $d\tau_0(\pi_0)/d\pi_0 > 1$, i.e., when the gas directly in front of the tube is supplied with heat when the pressure increases ($v_0 > 0$) and releases heat when the pressure drops ($v_0 < 0$). Here, a negative pressure gradient ($\partial\tau_{e1}/\partial x < 0$) corresponds to $v_0 > 0$, while a positive pressure gradient ($\partial\tau_{e1}/\partial x > 0$) corresponds to $v_0 < 0$. At the same time, $\partial\tau_{e1}/\partial\pi_0 > 1$.

For $x \ge u_d(\pi_0)$, the flow is in thermal equilibrium, provided the gas particles in the tube are in thermal equilibrium at the initial pressure.

The curves in Fig. 2 illustrate the aforesaid characteristics of the temperature field in nonstationary gas flows typical of thermal machines. The curves were plotted for $n = 1.66$, $v_0 = \text{const}$, $\partial\tau_s/\partial x = 0$, and $\tau_s(0) - \tau_0(\pi_s) = 0$. The initial temperature distribution in the tube ($p/p_s = 1$) and the temperature distributions that correspond to $u_d/n|v_0| = 0.7$ and to various (but constant for each curve) values of $\psi = d\tau_0(\pi_0)/d\pi_0$ are given on the graphs.

It is obvious that all relations obtained are applicable in the case where the gas particles participate in a polytropic process, in which case, n is the polytropic index.

NOTATION

p, ρ, T	are the pressure, density, and temperature of the gas, respectively;
R	is the gas content;
n	is an adiabatic exponent;
x	is the Euler coordinate;
u	is the coordinate of a fixed gas particle;
u_d	is the coordinate of the interface between the gas initially contained in the tube and the supplied gas;
t	is time;
$w = du/dt$;	
$\pi = \ln p$;	
$\tau = [n/(n-1)] \ln T$	
$v = w(d\pi_0/dt)^{-1}$;	
ξ	is the auxiliary variable of integration;
C_1, C_2	are independent integrals of the system of characteristic equations;
l_t	is the tube length;

*In order to determine the values of $\partial\tau_e/\partial\pi_0$ that correspond to given values of $\partial\tau_e/\partial x$, it is advantageous to use expression (11).

a is the speed of sound in the gas;
D/dt is the individual (substantive) derivative.

Subscripts

s represents initial state;
0 represents flow characteristics at the cross section $x = 0$;
e represents flow characteristics for $\partial\pi/\partial x = 0$.

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